shell to a system of linear algebraic equations by the Ritz method.

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ON THE CONTACT PROBLEM FOR A HALF-PLANE WITH FINITE ELASTIC REINFORCEMENT

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The state of stress of an elastic rod of finite length not acted on by bending moments and fastened to a semi-infinite plate is considered. The problem has already been investigated [1–3], but only for the simpler case where the load is applied to the rod ends. The present paper concerns the case where the force is applied to the center of the rod. The case where a heat source or a thermoelastic deformation center [4] is present at some point of the elastic half-plane is also considered.

As in the aforementioned studies, the problem is stated in the form of a Prandtl integro-differential equation; methods for solving the latter are the subject of an extensive literature [1, 5, 6]. The first step is to reduce the problem to a system of linear algebraic equations in some way. Some authors either give relatively complex general formulas for calculating the system coefficients in any approximation or no formulas at all. Other works contain quite simple formulas (as in the Multhopp method) but require recomputation of all the coefficients in each successive approximation.

The method of orthogonal polynomials [7] employed here quickly reduces the problem to an infinite system. This yields a very simple formula for the coefficients (in the case of a constant cross section), and affords a ready means of establishing the regularity and quasi-regularity of the system.

The paper concludes with an extension of the proposed method to rods of variable cross section. Cases where the cross section varies according to a linear or elliptic law are considered. The infinite system has a simple exact solution in the latter case.

1. Let a rod with the transverse cross-sectional area $F(x)$ be welded (cemented) over a finite segment $|x| \leqslant a$ of a semi-infinite plate (half-plane) of thickness h. The contact stresses $\tau(x)$ which arise along the line of contact between the rod and half-plane can be determined from the Prandtl integro-differential equation [1–3]

$$\int_{-a}^{x} \tau(t)\, dt + \frac{2E_1 F(x)}{\pi E_2 h} \int_{-a}^{a} \frac{\tau(t)}{x-t}\, dt = f(x) \tag{1.1}$$

where E_1, E_2 are the elastic moduli of the rod and half-plane, respectively, and $f(x)$ is a function which depends on the load.

Denoting the resultant of all the forces applied to the rod by R, we can write the self-evident equation

$$h \int_{-a}^{x} \tau(t)\, dt = \frac{h}{2} \int_{-a}^{a} \operatorname{sign}(x-t)\, \tau(t)\, dt + \frac{R}{2} \tag{1.2}$$

Let us set $F(x) = F(0)\, \psi(x/a)$, where $\psi(x/a) > 0$ for $|x| < a$. Substituting (1.2) into (1.1) and converting to the dimensionless coordinates $x^* = x/a$ and $t^* = t/a$ (from now on the asterisks will be omitted and the old symbols retained), we obtain

$$\int_{-1}^{1} \left[\frac{1}{2} \operatorname{sign}(x-t) + \frac{1}{\pi\lambda} \frac{\psi(x)}{x-t}\right] \tau(t)\, dt = g(x) \tag{1.3}$$

$$(\lambda = E_2 ah / 2E_1 F(0), \quad g(x) = [2hf(x) - R] / 2ah)$$

The solution of Eq. (1.3) must be subjected to the condition

$$\int_{-1}^{1} \tau(x)\, dx = \frac{R}{ah} \tag{1.4}$$

Let us express the required function $\tau(x)$ as a series in Chebyshev polynomials of the first kind,

$$\tau(x) = \frac{1}{\sqrt{1-x^2}} \sum_{m=0}^{\infty} X_m T_m(x) \tag{1.5}$$

then substitute (1.5) into (1.3), and apply the formulas

$$\int_{-1}^{1} \frac{T_m(t)\, dt}{(x-t)\sqrt{1-t^2}} \begin{cases} 0 & (m=0) \\ -\pi U_{m-1}(x) & (m = 1, 2, \ldots) \end{cases} \tag{1.6}$$

$$-\frac{1}{2}\int_{-1}^{1}\operatorname{sign}(x-t)\frac{T_m(t)\,dt}{\sqrt{1-t^2}}=\begin{cases}\arcsin x & (m=0)\\ -m^{-1}\sqrt{1-x^2}\,U_{m-1}(x) & (m=1,2,\ldots)\end{cases} \qquad \text{(cont.)}$$

($U_m(x)$ is a Chebyshev polynomial of the second kind)

Multiplying by $\sqrt{1-x^2}\;U_k(x)\,/\,\psi(x)$ and integrating over the interval $(-1,\ 1)$ with the aid of formula 7.343(2) of [8], we obtain the infinite system

$$\frac{\pi}{2\lambda}X_{k+1}+\sum_{m=1}^{\infty}X_m B_{m-1,k}=-B_{-1,k}X_0-b_k \qquad (k=0,1,\ldots) \qquad (1.7)$$

where

$$B_{m-1,k}=\frac{1}{m}\int_{-1}^{1}\frac{(1-x^2)}{\psi(x)}U_{m-1}(x)\,U_k(x)\,dx \qquad (m,k=0,1,\ldots) \qquad (1.8)$$

$$b_k=\int_{-1}^{1}\frac{g(x)}{\psi(x)}\sqrt{1-x^2}\,U_k(x)\,dx \qquad (k=0,1,\ldots) \qquad (1.9)$$

The first formula of (1.6) appears in [8] (p. 847). The second formula follows from the more general relation given in [9]. By virtue of (1.6), $-m^{-1}\sqrt{1-x^2}\,U_{m-1}(x)$ for $m=0$ in (1.8) must be replaced by $\arcsin x$.

Let us consider in more detail the case where the rod cross section is constant, i.e. where $F(x)=F(0)$ and $\psi(x)=1$. It is convenient to express the function $\tau(x)$ as a sum of two terms, $\tau(x)=\tau^+(x)+\tau^-(x)$, where $\tau^+(x)$ and $\tau^-(x)$ are the even and odd parts of the function $\tau(x)$, respectively. Then

$$\tau^+(x)=\frac{1}{\sqrt{1-x^2}}\sum_{m=0}^{\infty}X_{2m}T_{2m}(x) \qquad (1.10)$$

and the infinite system is of the form

$$\frac{\pi}{2\lambda}X_{2k+2}+\sum_{m=1}^{\infty}X_{2m}B_{2m-1,\,2k+1}=-B_{-1,\,2k+1}X_0-b_{2k+1} \qquad (k=1,0,\ldots) \qquad (1.11)$$

$$B_{2m-1,\,2k+1}=-\frac{8(k+1)}{[4m^2-(2k+1)^2]\,[4m^2-(2k+3)^2]} \qquad (m,k=0,1,\ldots) \qquad (1.12)$$

$$b_{2k+1}=\int_{-1}^{1}g(x)\sqrt{1-x^2}\,U_{2k+1}(x)\,dx \qquad (k=0,1,\ldots) \qquad (1.13)$$

Substituting (1.10) into (1.4), we find that $X_0=R\,/\,\pi ha$.

In obtaining the coefficients $B_{2m-1,\,2k+1}$ we use the formulas

$$\int_{-1}^{1}\arcsin x\,\sqrt{1-x^2}\,U_k(x)\,dx=\sin^2\frac{k\pi}{2}\,\frac{4(k+1)}{k^2(k+2)^2} \qquad (k=0,1,2,\ldots) \qquad (1.14)$$

$$\int_{-1}^{1}(1-x)^2\,U_m(x)\,U_k(x)\,dx=-\frac{4(k+1)(m+1)}{[(m+1)^2-k^2]\,[(m+1)^2-(k+2)^2]}\cos^2\frac{(m+k)}{2}\pi \qquad (m,k=0,1,\ldots) \qquad (1.15)$$

which are easy to verify by converting to the new variable $\gamma=\arccos x$ in the integrals and making use of formulas 10.11(2) of [10].

Similar results are obtainable in the case where $\tau(x)=\tau^-(x)$. They can be obtained by replacing $2m-1$ by $2m$ and $2k+1$ by $2k$ in formulas (1.10)–(1.13). In addition, we must set $X_0 = 0$ in system (1.11).

Now let us investigate system (1.11). We begin by showing that

$$\beta_k = \frac{2\lambda}{\pi} \sum_{m=1}^{\infty} |B_{2m-1,\,2k+1}| \to 0 \quad \text{as} \quad k \to \infty \tag{1.16}$$

i.e. that system (1.11) is quasiregular for all values of the parameter λ $(0 \leqslant \lambda < \infty)$.

First of all we note that the coefficients $B_{2m-1,2k+1} < 0$ for all m and k except in the case $m = k+1$. Isolating this term, we rewrite (1.16) as

$$\beta_k = \frac{2\lambda}{\pi} \left[\frac{8(k+1)}{(4k+3)(4k+5)} - \sum_{m=1}^{\infty}{}' B_{2m-1,\,2k+1} \right] \tag{1.17}$$

where the prime means that the sum does not include the term for $m = k+1$. Adding the missing term to the sum in (1.17) and then expressing the coefficient $B_{2m-1,\,2k+1}$ as a sum of two terms, we obtain two series summable with the aid of formula 1.421(3) of [8]. Carrying out the necessary operations, we find that Eq. (1.17) becomes

$$\beta_k = \frac{2\lambda}{\pi} \left[\frac{16(k+1)}{(4k+3)(4k+5)} - \frac{4(k+1)}{(2k+1)^2(2k+3)^2} \right] \qquad (k = 0, 1, \ldots) \tag{1.18}$$

so that condition (1.16) is valid for all $0 \leqslant \lambda < \infty$.

System (1.11) is regular if $\beta_k < 1$ for all k $(k = 0, 1, \ldots)$. We infer from (1.18) that the maximum value of β_k occurs for $k = 0$. The inequality $\beta_k \leqslant \beta_0 = 1.24\,\lambda/\pi$ is then valid for all k $(k = 0, 1, \ldots)$, which in turn implies that system (1.11) is regular for $\lambda < 2.5$. Thus, the regularity condition for infinite system (1.11) is broader than that of [1], namely $\lambda < 1.5$. Similar reasoning for the case $\tau(x) = \tau^-(x)$ yields the result $\lambda < 1.1$.

Let us consider the calculation of the coefficients b_{2k} and b_{2k+1} for certain particular cases of rod loading. Let the rod be acted on by two concentrated forces, each of magnitude $0.5P$, applied at the points $x = \pm\xi$ and directed along the x-axis. In this case we have

$$\tau(x) = \tau^+(x), \quad g(x) = \frac{P}{2ah}[e(x+\xi) + e(x-\xi) - 1], \quad b_{2k+1} = \frac{P}{2ah}\sqrt{1-\xi^2} \times \left[\frac{U_{2k}(\xi)}{2k+1} - \frac{U_{2k+2}(\xi)}{2k+3} \right] \tag{1.19}$$

where $e(x)$ is a Heaviside unit function. If $\xi = 1$ (the forces are applied to the rod ends) then $b_{2k+1} = 0$ $(k = 0, 1, 2, \ldots)$. If $\xi = 0$ (i.e. if a force of magnitude P is applied to the center of the rod) the coefficients b_{2k+1} are given by

$$b_{2k+1} = (-1)^k \frac{2P}{ah} \frac{(k+1)}{(2k+1)(2k+3)} \qquad (k=0,1,\ldots)$$

To find the coefficients b_{2k+1} (1.13) we must convert to the new variable $\gamma = \arccos x$ as above.

If the forces are directed in different directions, then $\tau(x) = \tau^-(x)$ and we obtain a formula similar to (1.19) for the coefficients b_{2k}.

If $\xi = 1$

$$b_0 = -\pi P/4ah, \quad b_{2k} = 0 \qquad (k=1,2,\ldots)$$

Computations show that the matrix of coefficients of system (1.11) and of the analogous system for $\tau = \tau^-$ has essentially diagonal dominance, which is a property very convenient for computational purposes. On the other hand, it indicates that the decrease of the unknown (required) coefficients is approximately the same as that of the coefficients of the right side. In addition, the decrease of the required coefficients also depends markedly on the parameter λ. For example, $\lambda = 1, 1.5, 5$ in the case $\xi = \pm 1$ required three, five, and eight approximations, respectively, for the determination of σ or τ. The number of approximations was such that the σ or τ in the mth approximation differed from the σ of τ in the ($m + 1$)-th approximation by not more than 5%.

For $\xi \neq \pm 1$ the coefficients b_k are Fourier coefficients in Chebyshev polynomials of a discontinuous function and therefore decrease gradually. Moreover, according to Melan's exact solution [11] for an infinite rod, the tangential stresses under the applied load increase without limit. Hence, the solution given by series (1.5) near the force can differ from the exact solution by an arbitrary amount. These difficulties in the case $\xi = 0$ are eliminated in the next section.

2. Let the concentrated force P be applied to the rod center (Fig. 1a). This problem can be solved by the superposition of Problems (b) and (c) (Fig. 1b and c). From now on we use $\tau(x)$, $\tau_1(x)$ and $\tau_2(x)$ (omitting the superscript + which indicates the evenness of the function), to denote the contact stresses which arise along the line of contact between the rod and plate in Problems (a), (b), and (c) (Fig. 1a, b, and c), respectively. In Problem (c) the rod ends are acted on by the stresses $\sigma_1 = | \sigma_x(\pm a) |$ which arise in the cross sections $x = \pm a$ of an infinite rod (Fig. 1b).

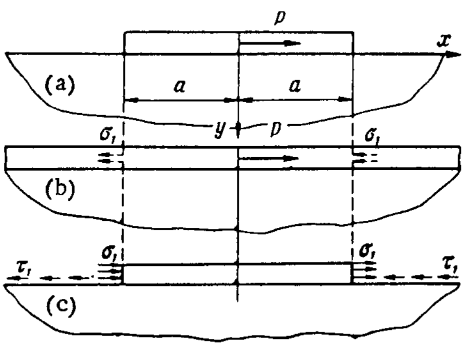

Fig. 1

Melan [11] gives the following solution of Problem (b):

$$\tau_1(x) = \frac{\lambda P}{\pi a h} \int_0^\infty \frac{\cos \alpha x}{\lambda + \alpha} d\alpha = -\frac{\lambda P}{\pi a h} (\sin \lambda x \operatorname{si} \lambda x + \cos \lambda x \operatorname{ci} \lambda x) \tag{2.1}$$

where $\operatorname{si}(y)$ and $\operatorname{ci}(y)$ are the integral sine and cosine, respectively. The functions occurring in (2.1) have been tabulated, and construction of the curve of $\tau_1(x)$ is not difficult. Problem (c) can be solved by the method described in the preceding section. In this case the coefficients b_{2k+1} are given by the formula

$$b_{2k+1} = \frac{1}{\pi \lambda} \int_{-1}^{1} \sqrt{1 - x^2}\, U_{2k+1}(x)\, dx \left(\int_{-\infty}^{\infty} - \int_{-1}^{1} \right) \frac{\tau_1(t)}{x - t} dt$$

Substituting (2.1) into the latter formula, reversing the order of integration, and applying the relations

$$\int_0^\infty \frac{J_n(x)}{\lambda + x} dx = \frac{(-1)^{n+1}}{2} [S_n(\lambda) + \pi E_n(\lambda) + \pi N_n(\lambda)] = \int_0^\infty e^{-nx - \lambda \operatorname{sh} x} dx \qquad (n = 0, 1, \ldots, \lambda > 0) \tag{2.2}$$

$$\int_{-1}^{1} \cos \alpha x T_{2k}(x)dx = -2\acute{c}_{0,2k}J_0(\alpha) - 4\sum_{n=1}^{\infty}(-1)^n c_{2n,2k} J_{2n}(\alpha) \qquad (2.3)$$

as well as formulas 3.722(8), 7.344(2) and 6.561(14) of [8], we obtain

$$b_{2k+1} = \frac{P}{ah}\{(-1)^k \lambda^{-1}(k+1)[(k+1)^{-1} + S_{2k+2}(\lambda) + \pi E_{2k+2}(\lambda) +$$

$$+ \pi N_{2k+2}(\lambda)] + c_{0,\,2k+2}[E_0(\lambda) + N_0(\lambda)] + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n c_{2n,\,2k+2}[S_{2n}(\lambda) +$$

$$+ \pi E_{2n}(\lambda) + \pi N_{2n}(\lambda)]\} \qquad (2.4)$$

Here $S_n(x)$ are Schläfli polynomials ([8], p. 1003), $E_n(x)$ is a Weber function ([8], p. 1002), $N_n(x)$ is a Neumann function, $J_n(x)$ is a Bessel function of the first kind, and finally

$$c_{2n,\,2k} = \frac{4(n^2+k^2)-1}{[(2n+1)^2-4k^2][(2n-1)^2-4k^2]} \qquad (n, k = 0, 1, \ldots)$$

Relation (2.2) is readily obtainable by using the identity

$$\int_0^{\infty} \frac{J_n(x)}{\lambda + x}dx = \int_0^{\infty} J_n(x)\,dx \int_0^{\infty} e^{-(\lambda+x)t}dt, \qquad \lambda > 0$$

reversing the order of integration, and applying formulas 6.611(1), 3.374(2) of [8], and 7.12(47) of [10]. To obtain relation (2.3) we must replace $\cos \alpha x$ by its series in Chebyshev polynomials of the first kind and integrating term by term. The series occurring in (2.3) and (2.4) converge quite rapidly.

The contact stresses in Problem (a) are $\tau(x) = \tau_1(x) + \tau_2(x)$, where $\tau_1(x)$ and $\tau_2(x)$ are given by formulas (1.10) and (2.1). It is easy to show that the normal stresses in the rod are given by the formula

$$\sigma(x) = -\sigma_1 + \frac{ah}{F(0)}\left[X_0\left(\arcsin x + \frac{\pi}{2}\right) - \sqrt{1-x^2}\sum_{m=1}^{\infty}\frac{X_{2m}}{2m}U_{2m-1}(x)\right] +$$

$$+ \frac{P}{\pi F(0)}[\cos \lambda x \operatorname{si} \lambda x - \sin \lambda x \operatorname{ci} \lambda x]$$

The last term of this expression represents the normal stresses in the Melan problem [11].

The greatest difficulties arise in calculating the coefficients b_{2k+1}. However, a practically acceptable solution is obtainable in some cases without their computation. In other words, the solution of Problem (a) can be obtained by superimposing Problems (b) and (c) without allowance for the tangential stresses $\tau_1(x)$ applied outside the segment $(-a, a)$ in Problem (c). This fact is illustrated in Table 1 which gives values of σ and τ computed with allowance for the indicated coefficients (the first and third rows) and without allowance for them (the second and fourth rows) in the case $\lambda = 1.0$. The tabulated values of τ represent fractions of P/ah and those of σ fractions of $P/F(0)$. We see from the Table that the maximum disparity between the normal stresses computed with and without allowance for $\tau_1(x)$ outside the segment $(-a, a)$ is 12.7% and occurs near the ends of the rod, where these stresses have their minimum values. The disparity of the tangential stresses does not exceed 7.8%. The indicated errors decrease as the parameter λ increases, reaching 4.8% and 2.3%, respectively, for $\lambda = 5$. In contrast to

Sect. 1, the number of approximations was such that the σ or τ obtained in the mth approximation began to differ from the σ or τ in the $(m+1)$-th approximation in the fourth place only.

Table 1

0	x	0	0.1	0.2	0.3	0.4
1	σ	0.500	0.403	0.342	0.303	0.265
2		0.500	0.403	0.342	0.301	0.263
3	τ	∞	0.672	0.490	0.398	0.342
4		∞	0.675	0.496	0.406	0.355
5	σ	0.149	0.148	0.145	0.139	0.131
6	τ	0	0.0241	0.0472	0.0685	0.0879
7	σ	0.126	0.124	0.117	0.107	0.0948
8	τ	0	0.0446	0.0836	0.113	0.131

0	x	0.5	0.6	0.7	0.8	0.9	1.0
1	σ	0.233	0.203	0.174	0.143	0.103	0
2		0.229	0.197	0.166	0.132	0.0914	0
3	τ	0.307	0.292	0.294	0.336	0.481	∞
4		0.325	0.314	0.319	0.357	0.476	∞
5	σ	0.122	0.110	0.0963	0.0798	0.0578	0
6	τ	0.106	0.125	0.149	0.186	0.272	∞
7	σ	0.0812	0.0671	0.0532	0.0396	0.0256	0
8	τ	0.140	0.140	0.137	0.136	0.150	∞

3. Let us consider the following thermoelasticity problem. Suppose that a heat source of strength W_0 acts at the point $x = 0$, $y = \eta$ in the half-plane; the boundary $x = 0$ is maintained at the temperature $t^\circ = 0$ and there is no heat transfer in the planes bounding the plate. Writing out the conditions of the compatible deformation of the plane and rod for $|x| \leqslant a$ and making use of expressions (25.28) of [4], we reduce the problem to Eq. (1.3) where, however,

$$g(x) = \frac{W_0 E_2 \alpha_t}{2\pi\theta\lambda} \frac{\eta^2}{\eta^2 + x^2}, \qquad R = 0$$

Here and below α_t and θ denote the coefficient of linear expansion and the thermal conductivity of the plate, respectively.

In this case $\tau(x) = \tau^-(x)$, so that the system contains odd unknowns only and the coefficient of the right side must be computed from the formula

$$b_{2k} = (-1)^k \frac{W_0 E_2 \alpha_t \eta}{2\theta\lambda} \left(\sqrt{1+\eta^2} - \eta\right)^{2k+1}$$

The results are similar in the case where a thermoelastic deformation center ([4], p. 225) is situated at the point $x = 0$, $y = \eta$,

$$g(x) = \frac{2E_1 F(0)\alpha_t}{\pi a^3 h} \frac{(\eta^2 - x^2)}{(\eta^2 + x^2)^2}$$

$$b_{2k} = \frac{2E_1 F(0)\alpha_t}{a^3 h} (-1)^k (2k+1) \frac{\left(\sqrt{1+\eta^2} - \eta\right)^{2k+1}}{\sqrt{1+\eta^2}}$$

The above formulas for b_{2k} are obtainable from the expansions

$$\frac{1}{\eta^2+x^2}=\frac{2}{\eta}\sum_{m=0}^{\infty}(-1)^m(\sqrt{1+\eta^2}-\eta)^{2m+1}U_{2m}(x),\quad \eta^2>0$$

$$\frac{\eta^2-x^2}{(\eta^2+x^2)^2}=\frac{2}{\sqrt{1+\eta^2}}\sum_{m=0}(-1)^m(2m+1)(\sqrt{1+\eta^2}-\eta)^{2m+1}U_{2m}(x),\quad \eta^2>0$$

which follow from the relations

$$\int_{-1}^{1}\frac{\sqrt{1-x^2}}{\eta^2+x^2}U_{2m}(x)\,dx=\frac{\pi}{\eta}\cos\frac{m\pi}{2}(\sqrt{1+\eta^2}-\eta)^{m+1} \tag{3.1}$$

$$\int_{-1}^{1}\frac{(\eta^2-x^2)}{(\eta^2+x^2)^2}\sqrt{1-x^2}\,U_m(x)\,dx=\pi(m+1)\cos\frac{m\pi}{2}\,\frac{(\sqrt{1+\eta^2}-\eta)^{m+1}}{\sqrt{1+\eta^2}} \tag{3.2}$$

$$(\eta^2>0;\quad m=0,1,\ldots)$$

These formulas are easy to verify by replacing the functions

$$(\eta^2+x^2)^{-1}\ \text{ and }\ (\eta^2-x^2)/(\eta^2+x^2)^2$$

in (3.1) and (3.2) by their integral representations as given by formulas 3.893(2) and 3.944(12) of [8], reversing the order of integration, converting to the variable $\gamma=\arccos x$, and applying formulas 3.715(18), 6.611(1), 6.623(3) of [8] and 10.11(2) of [10].

Numerical data for the case $\lambda=\eta=1$ are given in Table 1. Rows 5 and 6 apply to the case where a heat source acts at the point $x=0,\ y=\eta=1$ of the half-plane; rows 7 and 8 apply to the case where a thermoelastic deformation center is situated at this point. In the case of a heat source the values of σ and τ given in the Table must be multiplied by $W_0E_2\alpha_t ah/20F(0)$ and $W_0E_2\alpha_t/20$, respectively; the value in the case of a thermoelastic deformation center must be multiplied by $2E_2\alpha_t h/aF(0)$ and $2E_2\alpha_t/a^2$. Computation of τ required five approximations; computation of σ required two. The number of approximations required was determined by the criterion of Sect. 1. The number of approximations must be increased with increasing λ. For example, for $\lambda=5$ computation of τ requires eight approximations and computation of σ two approximations. The number of approximations also depends (but not quite so strongly) on η.

4. Knowing $\tau(x)$ and the equilibrium conditions, we can obtain an expression for the normal stresses acting in the rod. Some authors [3] prefer to begin with the determination of the function $\varphi(x)$ which represents the reduction of the axial force in the rod. With this approach it is convenient to find the function directly from the equation

$$\frac{\varphi(x)}{\psi(x)}+\frac{1}{\pi\lambda}\int_{-1}^{1}\frac{\varphi'(t)}{x-t}dt=\frac{g(x)}{\psi(x)}$$

where $\psi(x)$ has the same meaning as above. In wing theory the function $\varphi(x)$ represents the circulation of the air flow around the wing and is usually denoted by $\Gamma(x)$; the function $\psi(x)$ describes the wing contour. We know that the function $\varphi(x)$ must satisfy the conditions $\varphi(-1)=\varphi(1)=0$; for this reason we seek it in the form

$$\varphi(x)=\sqrt{1-x^2}\sum_{m=0}^{\infty}Y_mU_m(x),\qquad \varphi'(x)=\frac{-1}{\sqrt{1-x^2}}\sum_{m=0}^{\infty}(m+1)Y_mT_{m+1}(x)$$

As in the case of Eq. (1.3), we arrive at the infinite system

$$\frac{\pi}{2\lambda}(k+1)Y_k+\sum_{m=0}^{\infty}Y_mC_{m,k}=b_k \qquad (k=0,1,\ldots) \tag{4.1}$$

$$C_{m,k}=\int_{-1}^{1}\frac{(1-x^2)}{\psi(x)}U_m(x)U_k(x)\,dx \qquad (m,k=0,1,\ldots) \tag{4.2}$$

and the coefficients b_k are given by formula (1.9).

Let $\psi(x)=1$. In this case the coefficients $C_{m,k}$ can be determined from formula (1.15). It is convenient to break down the problem into even and odd parts as above. In the even case we have

$$\varphi^+(x)=\sqrt{1-x^2}\sum_{m=0}^{\infty}Y_{2m}U_{2m}(x)$$

and system (4.1) becomes

$$\frac{\pi}{2\lambda}Y_{2k}(2k+1)+\sum_{m=0}^{\infty}Y_{2m}C_{2m,2k}=b_{2k} \qquad (k=0,1,\ldots) \tag{4.3}$$

The coefficients $C_{2m,2k}$ are given by the expression (*)

$$C_{2m,2k}=-\frac{4(2k+1)(2m+1)}{[(2m+1)^2-4k^2][(2m+1)^2-(2k+2)^2]} \qquad (m,k=0,1,\ldots)$$

We arrive at similar results in the case of an odd function $\varphi(x)$

$$\varphi^-(x)=\sqrt{1-x^2}\sum_{m=0}^{\infty}Y_{2m+1}U_{2m+1}(x)$$

$$\frac{\pi}{2\lambda}Y_{2k+1}(2k+2)+\sum_{m=0}^{\infty}Y_{2m+1}C_{2m+1,2k+1}=b_{2k+1} \qquad (k=0,1,\ldots) \tag{4.4}$$

$$C_{2m+1,2k+1}=-\frac{16(m+1)(k+1)}{[(2m+2)^2-(2k+1)^2][(2m+2)^2-(2k+3)^2]} \qquad (m,k=0,1,\ldots)$$

The results of Sect. 1 remain valid for systems (4.3) and (4.4), since the latter are reducible to the systems investigated there.

We can verify this for system (4.3) by introducing into it the new unknowns $X_{2k+1}=(2k+1)Y_{2k}$, whereupon it becomes the system for the case $\tau(x)=\tau^-(x)$. The same can be accomplished for system (4.4) by substituting in the new unknowns according to the formula $X_{2k+2}=(2k+2)Y_{2k+1}$ and replacing the subscript m by $m-1$. The resulting system coincides with (1.11).

In conclusion we consider certain cases where the rod cross section varies along its length. Let the cross section vary linearly, i.e. let $\psi(x)=1-\alpha x$ $(\alpha\leqslant 1)$. Replacing the $[\psi(x)]^{-1}$ in (1.8) by its expansion in Chebyshev polynomials of the second kind (see [12], p. 31) and integrating term by term, we arrive at the following expression for the coefficients $B_{m-1,k}$ $(m,k=0,1,\ldots)$:

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*) After submitting the present paper for publication we learned that a similar formula had been derived by V. V. Golubev (Trudy TsAGI №108, 1931) by a more roundabout method involving the use of trigonometric series.

$$B_{m-1,k} = -\frac{8p}{\alpha} \sum_{n=0}^{\infty} p^n \sin^2 \frac{m+k+n}{2} \pi \frac{(n+1)(k+1)}{[m^2-(n+k+2)^2][m^2-(n-k)^2]} \quad (4.5)$$

$$p = (1 - \sqrt{1-\alpha^2})/\alpha$$

The series in (4.5) converges rapidly, since it is always the case that $p \leqslant 1$.

As our second example we consider a rod whose cross section varies according to an elliptic law, i.e. the case $\psi(x) = \sqrt{1-x^2}$. Formulas (1.7)–(1.9) now give us

$$B_{m-1,k} = B_{k,k} = \frac{\pi}{2(k+1)}, \qquad X_{k+1} = -\frac{2\lambda(k+1)}{\pi(\lambda+k+1)}(B_{-1,k}X_0 + b_k) \qquad (m, k = 0, 1, \ldots)$$

In particular, if $g(x) = M\sqrt{1-x^2}$ (M = const) and $X_0 = 0$, we obtain the exact solution

$$\tau^-(x) = -\frac{\lambda M}{\lambda+1} \frac{x}{\sqrt{1-x^2}}$$

Similar operations in the case of system (4.4) yield

$$C_{m,k} = C_{k,k} = \frac{\pi}{2}, \qquad Y_k = \frac{2\lambda b_k}{\pi(\lambda+k+1)} \qquad (m, k = 0, 1, \ldots)$$

If $g(x) = M\sqrt{1-x^2}$ (M = const) we have

$$Y_0 = \frac{\lambda M}{\lambda+1}, \qquad \varphi^+(x) = \frac{\lambda M}{\lambda+1}\sqrt{1-x^2}$$

The latter formula is familiar to us from wing theory.

In general the function $[\psi(x)]^{-1}$ must be interpolated by means of Chebyshev polynomials of the second kind,

$$\frac{1}{\psi(x)} = \sum_{n=0}^{N} a_n U_n(x)$$

In this case formula (4.2) yields the following expression for the coefficients $C_{m,k}$:

$$C_{m,k} = -4(m+1) \sum_{n=0}^{N} \frac{a_n (n+k+1)\cos^2[(m+n+k)\pi/2]}{[(m+1)^2-(n-k)^2][(m+1)^2-(n+k+2)^2]} \qquad (m, k = 0, 1, \ldots)$$

The thoeory and applications of interpolation by orthogonal polynomials are presented in [13].

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LOCAL INHOMOGENEITIES IN AN ELASTIC MEDIUM

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A method for investigating the perturbations of the external field due to a system of local inhomogeneities (defects) in an elastic medium is proposed. The method is based on a certain special representation of Green's tensor for a medium with defects in terms of the interaction energy operator which is convenient for describing the asymptotic behavior of perturbed fields. If the defects are small compared with the distances between them this representation makes it possible to construct effective solutions and to find expressions for the energy and for the interaction forces between the defects.

Section 1 introduces the interaction energy operator and deals with the construction of the asymptotic representation of Green's tensor for a homogeneous medium containing a single defect. The procedure for calculating the coefficients of the expansion is presented in Sect. 2 by way of an example (an ellipsoidal inhomogeneity). Section 3 concerns the general case of interaction of a defect system. Section 4 contains sample calculations for two ellipsoidal inhomogeneities. An explicit expression for the asymptotic behavior of the interaction energy is derived and special cases considered.

1. We begin with the general scheme. Let L_0 be a linear operator associated with a known Green's function G_0 satisfying certain boundary conditions, namely $L_0G_0 = I$, where I is an identity operator. If L_1 is a perturbation of the operator L_0 such that there exists a Green's function G of the operator $L = L_0 + L_1$, we can show that G is given by the representation

$$G = G_0 - G_0PG_0 \tag{1.1}$$

where the operator P is defined by the expression

$$P = L_1 (L_1 + L_1G_0L_1)^{-1}L_1 \tag{1.2}$$

In fact, substituting P into (1.1) and applying L from the left side, we obtain